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Space–Time Decay for Solutions of Wave Equations

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IN MEMORY OF NORMAN LEVINSON

The L_6 norm in space–time of a solution of the Klein–Gordon equation in two space–time dimensions is bounded relative to the Lorentz-invariant Hilbert space norm; the L_p norms for $p \geqslant 6$ are bounded relative to certain similar larger Hilbert space norms, including the energy norm.

1. INTRODUCTION

An essential role in relativistic scattering theory is played by the suitable decay of the solutions of the "free" equation. For example, in the case of the Klein–Gordon equation, $\Box\varphi = m^2\varphi$, $m \neq 0$, in n space dimensions, sufficiently regular solutions decay at the rate

$$|\varphi(x, t)| = O(|t|^{-n/2}), \qquad |t| \to \infty \tag{*}$$

uniformly in x; and this estimate provides the starting point for a number of investigations of scattering for nonlinear equations, such as the equation $\Box\varphi = m^2\varphi + g\varphi^p$ (see [6] for early work in this direction, and [7] for references to further work).

Despite the great progress in scattering theory over the past decade and a half, there remain fundamental unsolved problems, for which a different type of decay estimate might prove fruitful. The estimate (*) is itself best possible for sufficiently regular solutions; if the decay is more rapid, φ must vanish identically. Decay rates in L_p, for values of p other than the value $p = \infty$ involved in (*), are useful and have been determined (see [5, 8]), but are not sufficiently rapid for many nonlinear problems.

Two different developments, apart from general considerations,

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suggest the introduction of a mixed space–time norm to measure decay. First, in the work of Morawetz and Strauss [4] on relativistic nonlinear scattering theory, in which particularly incisive results for a specific nonlinear equation is obtained, certain a priori space–time bounds for solutions of the nonlinear equation play a key role. Second, Hörmander in [2] has obtained an interesting precise space–time estimate for the Schrödinger equation in two space–time dimensions.

In this note I examine by a method due originally to Carleson and Sjölin [1] and developed further by Hörmander in [2], the space–time decay of free solutions of the Klein–Gordon equation in two space–time dimensions. It is shown in particular that any Klein–Gordon wave function of finite energy is in L_6 as a function on space–time. It is interesting that this conclusion is identical with that for the Schrödinger wave function of a free particle with L_2 initial data. This could hardly have been foreseen, there being an essential point of difference between the two cases. The Carleson–Sjölin–Hörmander method applies typically to integrals such as Fourier transforms of distributions supported by lower-dimensional compact varieties of curvature bounded away from zero. The Fourier transform of a Klein–Gordon wave function is however supported by a noncompact variety whose curvature tends to zero at infinity.

Energy estimates have the advantage of temporal invariance as well as applicability to nonlinear equations. In contrast, the L_q norms on the Cauchy data which are required to ensure decay of suitable L_p norms in space as $|t| \to \infty$ are not continuous (Littman [3] first noted related behavior) and are relatively difficult to estimate. It remains to be seen however whether estimates of the present type are valid for nonlinear wave equations.

2. The Klein–Gordon Equation in Two Dimensions

Consider the distributions φ on R^2 of the form

$$\varphi(x, t) = \int_{-\infty}^{\infty} e^{it(1+y^2)^{1/2}-ixy} f(y)\, dm(y),$$

where $dm(y) = (1 + y^2)^{-1/2}\, dy$, and f is a measurable function such that $\int |f(y)|^2\, dm(y) < \infty$. These distributions satisfy the differential equation

$$\square\varphi = \varphi \qquad (\square = \partial^2/\partial x^2 - \partial^2/\partial t^2)$$

(here the parameter m is taken equal to 1, which suffices for the purposes of this note by virtue of scale covariance). The totality **H** of all such distributions is well known to form a Hilbert space on which the natural action of the Lorentz group G:

$$\varphi(X) \to \varphi(T^{-1}X), \qquad T \in G, X = (x, t),$$

is unitary with respect to the inner product determined by the equation

$$\langle \varphi, \varphi \rangle = \int |f(y)|^2 \, dm(y).$$

The "energy" operator in this Hilbert space is that whose corresponding action on f is $f(y) \to (1 + y^2)^{1/2} f(y)$.

It is evident that in order to bound the norm of φ in $L_p(R^2)$ for some value of p, it suffices to treat its positive energy component $\varphi^+(x, t) = \int_0^\infty e^{it(1+y^2)^{1/2}-ixy} f(y)\, dm(y)$. Consideration may therefore (and will) be confined to those distributions φ for which $f(y) < 0$ for $y < 0$. It will also be no essential loss of generality to assume that f is of class C^∞ and of compact support. It follows that

$$\varphi(x, t)^2 = \int_0^\infty \int_0^\infty e^{itu_2 - ixu_1} f(y_1) f(y_1) \, dm(y_1) \, dm(y_2),$$

where

$$u_1 = y_1 + y_2, \qquad u_2 = (1 + y_1^2)^{1/2} + (1 + y_2^2)^{1/2}.$$

Making the change of variables $(y_1, y_2) \to (u_1, u_2)$ in the expression for $\varphi(x, t)^2$, it results that

$$\varphi(x, t)^2 = \int_2^\infty \int_0^\infty e^{itu_2 - ixu_1} F(u_1, u_2) \, du_1 \, du_2,$$

where

$$F(u_1, u_2) = f(y_1) f(y_2)(1 + y_1)^{-1/2}(1 + y_2)^{-1/2} J^{-1},$$

J being the Jacobian

$$J(u_1, u_2 / y_1, y_2) = y_2(1 + y_2^2)^{-1/2} - y_1(1 + y_1^2)^{-1/2},$$

To show that $\varphi \in L_s(R^2)$ for some value of s is to show that $\varphi^2 \in L_{s/2}(R^2)$, which by the Hausdorff–Young inequality is implied by a demonstration that $F \in L_p(R^2)$, where $1 \leqslant p \leqslant 2$ and p' $(= p/(p-1)) = s/2$. Consider therefore the integral

$$I_p = \iint |F(u_1, u_2)|^p \, du_1 \, du_2 ;$$

on making the reverse change of variables, $(u_1, u_2) \to (y_1, y_2)$, I_p may be written as

$$\iint |f(y_1) f(y_2)|^p [(1+y_1^2)(1+y_2^2)]^{-p-2} \, | J |^{1-p} \, dy_1 \, dy_2 .$$

This integral will be estimated in terms of the L_2-type, energy-related norm

$$\langle \varphi, \varphi \rangle_c = \int (1+y^2)^{c/2} \, |f(y)|^2 \, dm(y);$$

the corresponding Hilbert space will be denoted $\mathbf{H}_c$; for $c = 0$ and $c = 1$, these are respectively the relativistic and energy norms. Introducing the function

$$g(y) = f(y)(1+y^2)^{(c-1)/4},$$

for which $\| g \|_{L_2} = \| \varphi \|_c$, I_p may be expressed as

$$\iint |g(y_1) g(y_2)|^p [(1+y_1^2)(1+y_2^2)]^{-p/2-p(c-1)/4} \, | J |^{1-p} \, dy_1 \, dy_2 .$$

To show that I_p is finite, it suffices to consider sufficiently large values of y_1 and y_2, for which the following estimate is readily obtained

$$| J | \geqslant \tfrac{1}{2} \, | y_2^2 - y_1^2 | \, y_1^{-2} y_2^{-2}.$$

Thus

$$I_p \leqslant C \iint |g(y_1) g(y_2)|^p [(1+y_1^2)(1+y_2^2)]^{-(p/2)-p((c-1)/4)}$$
$$\times \, | y_1^2 y_2^2 (y_1^2 - y_2^2)^{-1} |^{p-1} \, dy_1 \, dy_2 .$$

Now set $u = y_1^2$, $v = y_2^2$, and define h by the equation $h(u) = g(u^{1/2})$. This change of variable gives

$$I_p \leqslant C \iint | h(u) \, h(v)|^p [(1+u)(1+v)]^{-p((c+1)/4)} (uv)^{p-\frac{3}{2}} \, | u - v |^{1-p} \, du \, dv.$$

Setting $k(u) = | h(u)|^p (1 + u)^{-p((c+1)/4)} u^{p-3/2}$, it follows that

$$I_p \leqslant C \iint k(u)\, k(v) \,|\, u - v \,|^{1-p}\, du\, dv.$$

By the classical fractional integration inequality (cf. [9]) the latter integral is bounded by $C(2 - p)^{-1} \| k \|_q^2$ for $p \in [1, 2)$, where $q = 2(3 - p)^{-1}$.

In order to estimate $\| k \|_q$, apply Hölder's inequality $| \int AB | \leqslant \| A \|_a \| B \|_b$ $(a^{-1} + b^{-1} = 1;\ a,\ b \geqslant 1)$ with $A = | h(u)|^{pq} u^{-d}$ (d to be fixed later), $B = (1 + u)^{-pq(c+1)/4} u^{q(p-3/2)+d}$, which are such that $k^q = AB$, and with $a = (3 - p)/p$, $b = (3 - p)/(3 - 2p)$, which are $\geqslant 1$ if $p \in [1, \frac{3}{2}]$, which is henceforth assumed.

With these values,

$$\int A^a = \int | h(u)|^2\, u^{-ad}\, du,$$

which equals $C \int | g(y)|^2\, dy = C\langle \varphi, \varphi \rangle_c$ provided that $ad = \frac{1}{2}$, or equivalently if $d = 2^{-1} p (3 - p)^{-1}$, which value for d is now fixed. The remaining integral to be estimated is then

$$B^b = \int (1 + u)^{-bpq(c+1)/4}\, u^{b[q(p-\frac{3}{2})+d]}\, du,$$

which is convergent if (and only if)

$$bpq(c + 1)/4 > b[q(p - \tfrac{3}{2}) + d] + 1.$$

This inequality reduces straightforwardly to the inequality $p(c + 1) > p$, which is satisfied if and only if $c > 0$, irrespective of the value of $p \in [1, \frac{3}{2}]$. Correspondingly, $s \in [6, \infty]$, so that $\varphi \in L_s(R^2)$ for all $s \geqslant 6$ provided $c > 0$. In case $c = 0$ and $p = \frac{3}{2}$,

$$\begin{aligned} \| k \|_q^q &= \int | h(u)|^2 (1 + u)^{-1/2}\, du \\ &= \int | g(y)|^2 (1 + y^2)^{-1/2}\, 2y\, dy \leqslant C \| g \|_{L_2} = C \| \varphi \|_0 . \end{aligned}$$

This establishes the

THEOREM. *Let φ denote a function on R^2 which satisfies the equation*

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2 \varphi = 0 \qquad (m > 0).$$

For $c > 0$ there exists a constant K such that

$$\| \varphi \|_{L_s(R^2)} \leqslant K \| \varphi \|_c$$

for all $s \in [6, \infty]$; for $c = 0$ this holds for $s = 6$.

On expressing the norm $\| \cdot \|_c$ in terms of the self-adjoint operator B in $L_2(R^1)$ given by the equation $B = (m^2 I - \partial^2/\partial x^2)^{1/2}$ (where $\partial/\partial x$ is taken in its usual formulation as a normal operator in $L_2(R^1)$), this result may be stated as the

COROLLARY. *With the notation of the theorem,*

$$\left(\int | \varphi(x, t)|^6 \, dx \, dt \leqslant x\right)^{1/6} \leqslant K(\| B^{1/2}\varphi(., t)\|_{L_2(R_1)} + \| B^{-1/2}(\partial/\partial t)(., t)\|_{L_2(R^1)}).$$

For $c > 0$ and $s \in [6, \infty]$,

$$\| \varphi \|_{L_s(R^2)} \leqslant K(\| B^{c+1/2}\varphi(., t)\|_{L_2(R^1)} + \| B^{c-1/2}\varphi(., t)\|_{L_2(R^1)}).$$

Remark 1. For the case $m = 0$, i.e., the wave equation, it is easily seen (via the Fubini theorem) that any solution in two space–time dimensions which has finite L_p norm, for some $p < \infty$, vanishes identically.

Remark 2. The coincidence that the wave functions lie in $L_6(R^2)$ both in the present case and that of the Schrödinger equation may be analyzed as follows. Given an abstract differential equation of the form $u''(t) + B^2 u = 0$, where $u(t)$ has values in a given Hilbert space $\mathbf{H}$ in which B is a given positive self-adjoint operator $> \epsilon I$ for some $\epsilon > 0$, there is an associated Schrödinger equation

$$\dot{\psi} = iB\psi, \qquad \psi = Cu(t) - iC^{-1}u'(t); \qquad C = B^{1/2}.$$

In the case of the Klein–Gordon equation, $\mathbf{H} = L_2(R^1)$, and B is as earlier; the resulting Schrödinger equation is then not that for a relativistic free particle, but is a nonlocal equation in space which is approximated by the latter equation via the approximation

$$(m^2 I - \Delta)^{1/2} \sim m - (\Delta/2m),$$

apart from the inessential additive constant m.

It can be seen from the observation that $C\mathbf{H}_c \subset \mathbf{H}_{c+1}$ $(c > 0)$, that it follows from the theorem that $\psi(\cdot)(\cdot) \in L_6(R^2)$, i.e., ψ is in L_6 as a function on space–time, provided $\psi(0)$ lies in the domain of B^δ for some $\delta > 0$. Conceivably the Fourier integral operators e^{itB} and $e^{it(m-\Delta/2m)}$ are sufficiently close in an appropriate space of such operators to permit a similar deduction for the case of the free-particle Schrödinger equation as a corollary. It would seem more difficult to reverse this type of procedure and obtain inferences about solutions of relativistic or other hyperbolic equations from results concerning solutions of Schrödinger equations.

Remark 3. The time t in the corollary is arbitrary in R^1, the right-hand side being t-independent as a consequence of the differential equation. The constant K is independent of s since it may be taken as the maximum of its values for $s = 6$ and $s = \infty$.

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